JGP - Vol. 5, n. 2, 1988

Construction of constant curvature punctured Riemann surfaces with particle-scattering interpretation

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Dedicated to I.M. Gelfand
on his 75th birthday

Abstract. *A class of punctured constant curvature Riemann surfaces, with boundary conditions similar to those of the Poincaré half plane, is constructed. It is shown to describe the scattering of particle-like objects in two Euclidian dimensions. The associated time delays and classical phase shifts are introduced and connected to the behaviour of the surfaces at their punctures. For each such surface, we conjecture that the time delays are partial derivatives of the phase shift. This type of relationship, already known to be correct in other scattering problems, leads to a general integrability condition concerning the behaviour of the metric in the neighbourhood of the punctures. The time delays are explicitly computed for three punctures, and the conjecture is verified. The result, reexpressed as a product of Riemann zeta-functions, exhibits an intringuing number-theoretic structure: a p-adic product formula holds and one of Ramanujan's identities applies. An ansatz is given for the corresponding exact quantum S-matrix. It is such that the integrability condition is replaced by a finite difference relation only involving the exact spectrum already derived, in the associated Liouville field theory, by Gervais and Neveu.*

INTRODUCTION

The developments of string theories and of conformally invariant field theories in two dimensions are interest both for theoretical physicists and mathematics. In this connec-

Key-Words: Punctured Riemann surfaces, particle-scattering.
1980 MSC: 30 F, 81 F 20.

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tion, our study of the related Liouville field theory led us to general results, concerning a particular class of two-dimensional manifolds with constant curvature, which we believe to be significant by themselves. We were motivated by the underlying dynamical problem related to string and conformal theories; and, so far, have described our work only from this viwpoint [1]. It seems appropriate to display it here again, in order to emphasize its geometrical aspect which may be of a wider interest.

The following general structure came out of our study [1], and will be displaid in the forthcoming sections. Consider a two-dimensional metric tensor g_{ab} defined on a Riemann surface. It has three independent components. By an appropriate change of parametrization (diffeomorphism), one can fix two of them. Choosing orthonormal coordinates, the metric becomes locally:

$$g = e^{\phi}\hat{g} \tag{1.1}$$

$$\hat{g} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.2}$$

Our aim is to determine a particular family of metrics with constant intrinsic curvature. In the conformal coordinates (1.1), this given:

$$4\,\frac{\partial^2\phi}{\partial z\partial z^*} = \text{const}\; e^{\phi} \tag{1.3}$$

Liouville introduced and solved this equation long ago. This is why the corresponding dynamics is called the Liouville field theory in theoretical physics. One may absorb the above constant by a shift of ϕ as we shall do from now on. The general solution of eq. (1.3) is rederived, in section 2, in a way that is suitable for our purpose.

We shall be interested in Riemann surfaces with boundaries that are piecewise differentiable. Within each differentiable arc segment, the boundary condition is that the normal derivative satisfies [2, 3]:

$$\partial_n\phi = -\sqrt{2}\,e^{\phi/2} + 2\,k_f \tag{1.4}$$

k_f is the curvature of the boundary in the «parameter space» (with metric $\hat{g}$). We shall only consider the case where by a conformal mapping we may choose z to vary in the upper complex half-plane. Thus the boundary will be made up with pieces of the real axis and $k_f = 0$. Since the geodesic curvature k_g is related to k_f by

$$k_f = e^{\phi/2}k_g + \frac{1}{2}\partial_n\phi \tag{1.5}$$

the boundary condition (1.4) simply states [4] that the geodesic curvature is a constant on each boundary

$$k_g = -1/\sqrt{2}. \tag{1.6}$$

Solving the Liouville equation (1.3) with the boundary conditions (1.4) is thus equivalent to finding a constant-curvature metric on the Riemann surface such that the geodesic curvature on each boundary is a constant. Remarkably, these equations came out from Polyakov's study of the string functional integral [5]and from the analysis of the associated string-dynamical boundary effects [2]. This motivated detailed studies of Gervais and Neveu [3, 6] and of ourselves [1, 7] some of which are the subjects of the present article. On each continuous arc segment of the boundary, the metric is singular at the boundary in such a way that, in its neighbourhood, the surface becomes isomorphic to the Poincaré half-plane. The boundary condition (1.4) is such that the metric may be continued through the boundary by inversions with respect to the continuous arc segments. This procedure, which is similar to the method of images, allows to extend the metric to a Riemann surface with punctures located where the original boundary is singular. Details are given in section 2.

The behaviour of the metric at the punctures is specified as follows: From equation (1.3), one easily derives:

$$\frac{\partial}{\partial z}\left(e^{\phi/2}\,\frac{\partial^2 e^{-\phi/2}}{\partial z^{*2}}\right) = \frac{\partial}{\partial z^*}\left(e^{\phi/2}\,\frac{\partial^2 e^{-\phi/2}}{\partial z^2}\right) = 0 \tag{1.7}$$

We may thus define to holomorphic functions $U(z)$ and $V(z)$ by:

$$e^{\phi/2}\,\frac{\partial^2 e^{-\phi/2}}{\partial z^2} \equiv U(z); \quad e^{\phi/2}\,\frac{\partial^2 e^{-\phi/2}}{\partial z^{*2}} \equiv V(z^*). \tag{1.8}$$

We shallo show, in sect. 2, that the boundary conditions (1.4) are such that V is deduced from U by inversion through the boundaries and that U is thereby continued as a meromorphic function with singualarities only at the punctures.

These features have an important dynamical meaning in the underlying conformal field theory. A Riemann surface with $P > 2$ punctures describes the interactions of P elementary excitations, called particles for short. A Riemann surface with two punctures decribes a free propagating particle. It is equivalent to the tube $-\infty < \xi < +\infty$, $0 < \eta < \pi$ plays the role of the time. The boundary is made up with the two lines $\eta = 0$ and $\eta = \pi$. The problem being invariant by time translation, the solution has a free parameter that specifies the origin of the time axis. It is rederived in section 3, following [1] and [3], in the case where it has double poles at the punctures. For $P > 2$, we have developed the scattering theory and introduced the corresponding physical quantities

(see sect. 4). The basic idea is as follows. In a small neighbourhood of one of the punctures, the detailed location of the others becomes unimportant and, to leading order, the metric takes the same form as if P were equal to 2. This small neighbourhood may be described by local ξ and η coordinates such that the puncture is reached for ξ going to infinity. In physical terms, this means that asymptotically, that is for large ξ, the particles do not interact. Such is the basic requirement of the scattering theory. We shall moreover show that the first correction to the behavior at $\xi = \infty$ is that the local-time translation invariance of the asymptotic local tube is broken. Its local-time hereby aquires a fixed origin whose location describes the leading effect of the other punctures. This corresponds to the standard physical time delay of classical scattering theory. There is one such time delay Δ_i for each puncture.

We have explicitely computed these time delays in the case of three punctures when U has double poles with residues specified by parameters λ_i. Our main explicit result, rederived in sect. 5 below, is the formula:

$$F_i(\lambda_1, \lambda_2, \lambda_3) \equiv e^{\lambda_i \Delta_i} = \tag{1.9}$$
$$= \frac{\zeta(1-\lambda_i)\zeta(-\lambda_i)}{\zeta(1+\lambda_i)\zeta(\lambda_i)} \prod_{\epsilon_1,\epsilon_2,\epsilon_3=\pm} \zeta\left(\frac{1}{2} + \sum_{k=1}^{3} \frac{\epsilon_k \lambda_k}{2}\right)^{\epsilon_i}$$

where ζ is the standard Riemann zeta-function. This formula has an interesting number-theoretic structure. First, it is well known that the zeta-function can be written as the following product over all prime numbers

$$\zeta(s) = \prod_{p=\text{prime}} \frac{1}{1-p^{-s}} \tag{1.10}$$

This shows that our function F_i can also be written as a product over all primes, which indicates that it has an underlying p-adic structure. Moreover, eq. (1.10) can be used [8] in order to derive the following equality due to Ramanujan [9, 10]

$$\frac{\zeta(s)\,\zeta(s-a)\,\zeta(s-b)\,\zeta(s-a-b)}{\zeta(2s-a-a)} = \sum_{n=1}^{\infty} n^{-s}\sigma_a(n)\,\sigma_b(n) \tag{1.11}$$

where

$$\sigma_c(n) = \sum_{d|n} d^c \tag{1.12}$$

denotes the sum of the cth powers of the divisors of n (including 1 and n). This particular combination of zeta-functions is precisely the one present in our expression (1.9) for F_i. Taking F_1 as an example, we make the identifications

$$s(\epsilon) = \frac{1}{2}(1 + \epsilon\lambda_1 + \lambda_2 + \lambda_3), \quad a = \lambda_2,\ b = \lambda_3 \tag{1.13}$$

and obtain

$$F_1(\lambda_1, \lambda_2, \lambda_3) = \prod_{\epsilon=\pm} \left(\frac{1}{\zeta\left(\frac{2m_1}{N}\right)} \sum_{n=1}^{\infty} n^{-s(\epsilon)} \sigma_a(n) \sigma_b(n) \right)^{\epsilon} \tag{1.14}$$

What is the structure of the metric near the puncture? The η-dependence of the two-puncture solution for double poles is through trigonometric functions of $\lambda\eta$, where λ characterizes the residues at the double pole. (see (3.9) below). After one has continued through the boundaries the parameter vary on the tube $-\infty < \xi < \infty, 0 < \eta < 2\pi$. For generic λ the two-puncture solution is thus multivalued with an infinite number of sheets. After mapping onto the sphere with two punctures, one sees that the metric has a limit at the puncture which depends upon the direction in a multivalued way. By the above scattering argument it follows that the behavior at each puncture is similar for the P-puncture solution as for the case $P = 2$, so that the surface with constant curvature is multisheeted at each puncture.

Physically, one regards the Liouville equation as decribing a two-dimensional continuous system. Equation (1.3) follows from the principle of least action if we introduce:

$$S = \frac{1}{16\pi\hbar} \int \mathrm{d}z\mathrm{d}z^* \left(2\frac{\partial\phi}{\partial z}\frac{\partial\phi}{\partial z^*} + e^{\phi} \right) \tag{1.15}$$

In classical dynamics, the factor in front of the action is irrelevant, and the spectrum is continuous. For our surfaces, this means that the parameters of the pole structures of U take continuous values, leading to an infinite number of sheets at the punctures. Quantum mechanically, the coefficent in front of the action defines the Planck constant $\hbar$ of the system, in term of which the levels are quantized. In conformal quantum field theories, of which the Louville theory is an example, a central role is plaid by the so-called rational theories. With the present boundary conditions they correspond to [6]:

$$\hbar = \frac{2N}{(N+1)^2}. \tag{1.16}$$

The spectrum of states becomes quantized [3, 6]. In particular, the parameters λ_j of the double poles only take discrete values, that is, [6]:

$$\lambda_i = 2\frac{m_i}{N}; \quad N, m_i, \text{integers}; |m_i| \leq \frac{N-1}{2} \tag{1.17}$$

The geometrical meaning of this quantization is that the corresponding surfaces now we have a finite number of sheets exactly equal to N.

Another general result concerns an integrability condition which is satisfied by the time delays Δ_i. It is a well-known fact of classical dynamics that the time delays are

derivatives of the classical phase shift with respect to the energy. In our case, and for double poles, the energy associated with the asymptotic metric at the ith puncture is a linear function of λ_i^2. The time delays, being partial derivatives of the same classical phase shift, must satisfy the integrability condition:

$$\frac{1}{\lambda_j}\frac{\partial \Delta_i}{\partial \lambda_j} = \frac{1}{\lambda_i}\frac{\partial \Delta_j}{\partial \lambda_i} \tag{1.18}$$

From the physical argument just recalled, we were led to postulate that this condition must be true for any number of punctures. We have indeed checked, on our explicit formula, that it is satisfied for $P = 3$. This type of relation which holds for the present class of surfaces with constant curvature is new to our knowledge. More details are given in section 6.

Finally an explicit formula is given, in section 7, for the exact S-matrix that the describes the quantum interaction whose classical limit corresponds to the time delays of formula (1.14). It is an ansatz which is such that the integrability condition (1.18) is replaced by a finite difference equation involving the values of λ_i that appear in the quantum spectrum (1.17). The fact that is possible together with the similarities of our formulae with the known exact S-matrix for the sine-Gordon theory, make it very likely that our formula is indeed exact. It enjoys number theoretic properties similar to the classical formula (1.14). Altogether, it displays an interesting connection between quantum mechanics, geometry, and number theory.

2. THE FRAME FOR THE GENERAL SOLUTION

Since we consider the case of Riemann surfaces which may be mapped onto the upper half plane, we take z in this domain, to begin with. Later on, we will extend the domain of z to a complete Riemann surface of certain functions. Let us first derive the general solution of the Liouville equation (1.3):

$$4\frac{\partial^2 \phi}{\partial z \partial z^*} = e^{\phi}. \tag{2.1}$$

PROPOSITION. *The solution of the Liouville equation is of the form*

$$e^{-\phi/2} = \frac{1}{2i\sqrt{2}}\left(\psi_1(z)\chi_2(z^*) - \psi_2(z)\chi_1(z^*)\right) \tag{2.2}$$

where $\psi_{1,2}(z)$ *and* $\chi_{1,2}(z^*)$ *are two independent solutions of the associated linear differential (Schrödinger) equation*

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}z^2} + U(z)\right)\psi_{1,2}(z) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}z^{*2}} + V(z^*)\right)\chi_{1,2}(z^*) = 0 \tag{2.3}$$

with Wronskians equal to one. The functions $U(z)$ *and* $V(z^*)$, *called potentials are defined by (1.8).*

Proof. Equation (1.8) may be written as:

$$\left(-\frac{\partial^2}{\partial z^2} + U(z)\right) e^{-\phi/2} = \left(-\frac{\partial^2}{\partial z^{*2}} + V(z^*)\right) e^{-\phi/2} = 0$$

Given the two solutions of the first equation (2.3), we may integrate the first partial differential equation, obtaining:

$$e^{-\phi/2} = \sum_{j=1,2} \psi_j(z) C_j$$

where the integration constants C_j only depend of z^*. Using the second partial differential equation, we see that the C_j 's are linear combination of the χ 's, since they satisfy the same differential equation. This linear combination may always be chosen so that (2.2) holds. Moreover one verifies that (2.2) satisfies (2.1) if the Wronskians of the functions ψ_1, ψ_2, and χ_1, χ_2 is equal to one. Q.E.D. ∎

It is convenient to introduce a two-component notation

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}; \qquad \chi = \begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} \tag{2.4}$$

and rewrite the solution (2.2) as

$$e^{-\phi/2} = \frac{1}{2i\sqrt{2}} \psi^T(z)\, \zeta\, \chi(z^*), \quad \zeta = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.5}$$

A basic property of the Liouville theory is its conformal invariance. It comes from the fact that the choice of conformal coordinates (1.1, 2) is preserved if we perform analytic transformations on z. From this viewpoint, the last formula expresses the decomposition of the metric into a sum of products of homomorphic and antiholomorphic half differentials.

Consider now the boundary condition. Since we have chosen to take $z = x + iy$ in the upper complex half plane, the boundary is the real axis, and hence $(k_f = 0)$. Eq. (1.4) reads

$$\partial_n e^{-\phi/2} = \partial_y e^{-\phi/2} = \frac{1}{\sqrt{2}} \quad \text{at } y = 0 \tag{2.6}$$

This may seem complicated, but formula (2.2) allows to implement this boundary condition directly on the function ψ and χ, as we next show. Obviously, the argument z^* of χ and V varies over the lower half plane and we are going to extend the physical upper plane to what we call the physical sheet, which is the domain where z or z^* lie

in the range of the integral (1.15) that defines the action. This latter region has cuts on the real axis. Indeed, we are dealing with the Riemann surface of ψ_i and χ_i as we will discuss later on in more detail. In implementing condition (2.6), one must take account of the cuts. When y tends to zero, z^* will not, in general tend to z, since the limit is on a different sheet of the Riemann surface.

The general situation will be that there are $P-1$ cuts. The branch points located at $z_j, j = 1, \ldots, P.$ will give the punctures of the surface described by the constant curvature metric, which we discussed in the introduction. We take them to be located on the real axis and the indices are chosen such that $z_j < z_{j+1}$. The physical sheet will be defined in such a way that the upper and lower half planes are connected through a specific portion of the real axis where there are no cuts. By convention, we take this segment to be between z_1 and z_2. It is convenient, in general, to label by j the segment of the real axis that lies between z_j and z_{j+1}, and to denote by $\bar{x}_j$ the corresponding limit of z^* as $\operatorname{Im} z \to 0$. With these conventions equation (2.6) leads to

$$\psi^{T\prime}(x)\zeta\chi(\bar{x}_j) - \psi^T(x)\zeta\chi^\prime(\bar{x}_j) = 2; \quad \bar{x}_j = \lim_{y\to 0} z^* \text{ on segment } j \tag{2.7}$$

Since there is no cut along the segment 1, we have $\bar{x}_1 = x$, and the boundary condition on this segment plays a special role which we now discuss.

PROPOSITION. *The potential V introduced in eq. (2.3) is the analytic continuation to the lower complex half plane of the potential U.*

Proof. Taking the derivative of eq. (2.7) (on the segment 1) with respect to $x = \bar{x}_1$ and making use of the Schödinger equation (2.3) we obtain

$$(U(x) - V(x))\,(\psi^T(x)\zeta\chi(x)) = 0$$

for $x \in (z_1, z_2) \cdot U(x)$ and $V(x)$ are thus equal on a segment of the real axis and the proposition follows from standard complex analysis. Q.E.D. ∎

V is the analytic continuation of U to the lower half plane through the particular segment considered. From now on, we do not make use of the function V any longer. Instead, U is considered as an analytic function defined on the physical sheet introduced above. From standard arguments about differential equations, it next follows that the ψ 's and the χ 's are analytic in the same region as U. We may therefore continue analytically the differential equation for χ in (2.3), to the upper half plane and thus substitute z for z^* there. This leads to

PROPOSITION. *$\psi_{1,2}(z)$ and $\chi_{1,2}(z)$ satisfy the same second order differential equation*

$$\left(-\frac{\mathrm{d}^2}{\mathrm{d}\,z^2} + U(z)\right)\psi_{1,2}(z) = \left(-\frac{\mathrm{d}^2}{\mathrm{d}\,z^{*2}} + U(z)\right)\chi_{1,2}(z) = 0 \tag{2.8}$$

and hence are related by a (constant) linear transformation

$$\begin{pmatrix}\chi_1\\ \chi_2\end{pmatrix} = T\begin{pmatrix}\psi_1\\ \psi_2\end{pmatrix}; \quad T_1 \equiv \begin{pmatrix}\alpha_1 & \beta_1\\ \gamma_1 & \delta_1\end{pmatrix} \tag{2.9}$$

■

The index 1 indicates that this condition is relevant to the number 1 of the real axis where there are no cuts.

The solution (2.3) takes the form

$$e^{-\phi/2} = \frac{1}{2i\sqrt{2}}\,\psi^T(z)\zeta T_1\,\psi(z^*) \tag{2.10}$$

Half of the degrees of freedom have been eliminated by the boundary condition. It is by now quite clear why the left member of (2.7) is a constant on the segment of the real axis considered: It is proportional to the Wronskian of the common Schrödinger equation (2.8). Substituting (2.9) into (2.7), we obtaind the condition:

$$Tr(T_1) \equiv \alpha_1 + \delta_1 = 2 \tag{2.11}$$

This terminates the discussion of the boundary condition on the segment number 1 where there are no cuts. Next we derive the

PROPOSITION. *For any segment j of the real axis, the boundary condition gives:*

$$\chi(\bar{x}_j) = T_j\psi(x),\ T_j \equiv \begin{pmatrix}\alpha_j & \beta_j\\ \gamma_j & \delta_j\end{pmatrix};\ \alpha_j\delta_j - \beta_j\gamma_j = 1 \tag{2.12}$$

and

$$Tr(T_j) \equiv \alpha_j + \delta_j = 2\,. \tag{2.13}$$

Proof. For $j = 1$, this was already proven. For $j \neq 1, \bar{x}_j$ differs from x. The left-hand side of (2.7) can be a constant only if there exists, nevertheless, a linear relation between $\chi(\bar{x}_j)$ and $\psi(x)$, so that the Wronskian appears again in (2.7). The T 's are unimodular since both χ 's and ψ 's have unit Wronskian. Equation (2.13) follows by substituting (2.12) into (2.7). Q.E.D. ■

Let us now come to our main result of this section:

THEOREM. *The boundary conditions can be fulfilled only if the potential is meromorphic in the whole complex z plane.*

Proof. Combining (2.9) with (2.12) for $j = 1$, one immediately gets:

$$\psi(x) = M_j \psi(\bar{x}_j); \; M = T_j^{-1} T_1 \tag{2.14}$$

By analytic continuation, it follows that the two solutions ψ_1 and ψ_2 of the Schrödinger equation (2.8) are transformed into linear combinations of themselves when one performs a rotation of 2π around any branch point z_j. Consequently, as is well known from the theory of linear differential equations [11], the potential U (which is the Schwarzian derivative of ψ_2/ψ_1) has only poles at the punctures z_j. Q.E.D. ∎

The different physical sectors of the theory will be specified by choosing the pole structure of U. As already mentioned in the introduction, a potential with P poles is to be considered as describing a scattering process involving P particles. In this discussion, the basic object, to start with, is the potential U. As shown in ref. [3] the Schrödinger equation (2.8) is one member of the Lax pair associated with the Liouville dynamics. Starting from U and constructing the classical solution by solving this equation is thus natural in the spirit of the inverse scattering method.

As a general comment we may stress that the present discussion is similar in spirit to standard string theories. On the one hand, the boundary condition plays the same role as the open string boundary condition: First, it is such that the left – and right – moving modes – that is the holomorphic and antiholomorphic half differentials of (2.5) – are correlated. Second, it leads to a natural extension of the solution by reflections about the boundaries. On the other hand, the general relationship between P-particle processes and Riemann surfaces with P punctures was originally introduced in string theories where it plays a key role. The variables z_j that are the locations of the poles of U are the analogues of the Koba-Nielsen variables of the string scattering amplitudes.

Finally, the boundary conditions we are using are such that, near the differentiable arc segments of the boudary, the metric diverges as the one of Poincaré half plane. This latter metric does in fact appear as a limiting case of the two-puncture metric to be discussed next.

3. TWO BRANCH POINTS – THE ONE-HIGHEST-WEIGHT SOLUTION

The simplest non-trivial case is a Riemann surface with one branch cut. The associated solutions with double poles were worked out in ref. [3]. We shall rederive them here again since they are essential for the following. The two branch points can be taken to lie at $z = 0$ and $z =$, the cut being located along the negative real axis. The Riemann surface of the solutions $\psi_i, i = 1, 2$, of the Schrödinger equation (2.8) has a branch cut from $z_1 = 0$ to $z_2 = \infty$ iff z_1 and z_2 are singular points of this differential equation (2.8). The two singularities are of the same type (i.e. they are double poles) iff the

potential U is of the form

$$U(z) = -\frac{1}{4}\,\frac{1-\lambda^2}{z^2} \tag{3.1}$$

λ is a parameter, whose physical meaning will be discussed below. It is clear that two independent solutions of the differential equations (2.8) with this potential are

$$\psi_1 = \frac{1}{\sqrt{\lambda}}\, z^{(1+\lambda)/2}; \quad \psi_2 = \frac{1}{\sqrt{\lambda}}\, z^{(1-\lambda)/2}. \tag{3.2}$$

Consider next the boundary condition on the negative real axis. With an obvious notation we have

$$\bar{x}_2 = x\, e^{-2\pi i}$$

The index 2 characterizes the fact that this equation holds for negative values of x (i.e. on the segment 2). The monodromy matrix is given by

$$\psi(x) = M\psi(\bar{x}_2); \quad M = -\begin{pmatrix} e^{i\pi\lambda} & 0 \\ 0 & e^{-i\pi\lambda} \end{pmatrix} \tag{3.3}$$

and the boundary conditions (2.13) yield the following relations for the elements of the matrix T

$$\alpha_1 + \delta_1 = -(\alpha_1 e^{-i\pi\lambda} + \delta_1 e^{i\pi\lambda}) = 2 \tag{3.4}$$

This and the unimodularity condition determine T almost completely:

$$\begin{aligned} \alpha_1 &= -i\frac{e^{i\pi\lambda/2}}{\sin(\pi\lambda/2)}; \quad \delta_1 = i\frac{e^{-i\pi\lambda/2}}{\sin(\pi\lambda/2)} \\ \beta_1 &= -ie^{\lambda\Delta}\ \operatorname{cotan}(\frac{\pi\lambda}{2}); \quad \gamma_1 = ie^{-1\lambda\delta}\ \operatorname{cotan}(\frac{\pi\lambda}{2}) \end{aligned} \tag{3.5}$$

There remains one undetermined constant Δ. The classical solution is obtained by substituing the result of the present computation in equation (2.10):

$$\begin{aligned} e^{-\phi/2} = \frac{1}{2\sqrt{2}}\,\frac{\sqrt{zz^*}}{\lambda}\Bigg(&\operatorname{cotan}\left(\frac{\pi\lambda}{2}\right)\left((zz^*e^{-2\Delta})^{\lambda/2} + (zz^*e^{-2\Delta})^{-\lambda/2}\right) \\ &+ \frac{1}{\sin(\pi\lambda/2)}\left(\left(e^{-i\pi}\frac{z}{z^*}\right)^{\lambda/2} + \left(e^{-i\pi}\frac{z}{z^*}\right)^{-\lambda/2}\right)\Bigg) \end{aligned} \tag{3.6}$$

This completes the determination of the solution with two punctures and hence of a one-parameter family of constant curvature metrics on a Riemann surface with one branch cut. The physical picture is seen by going to the cylinder through the mapping $z = \exp(\xi + i\eta)$. $-\infty < \xi < \infty$, $0 < \eta < 2\pi$. The upper half plane is mapped onto the strip $0 < \eta < \pi$. ξ, η are Euclidean coordinates and one may go to Minkovsky space by letting $\xi \to i\tau$. The latter variable is the physical time. The solutions (3.6) are periodic functions of τ and, hence, describe stationary states. In the particle language, they are thus isolated single states.

Concerning the viewpoint of conformally invariant field theory, let us remark that, as shown in ref. [3], the potential U is related to the energy-momentum tensor in such a way that the classical Virasoro generator read

$$L_n = -\frac{1}{2\hbar}\int \frac{dz}{2\pi i} z^{n+1} U(z) \tag{3.7}$$

and, in view of (3.1), the classical solution is such that

$$L_0 = \frac{1}{8\hbar}(1-\lambda^2); \qquad L_n = 0; \qquad n \neq 0 \tag{3.8}$$

When the Liouville theory is quantized, the L's become operators that satisfy the Virasoro algebra Vir. A highest-weight representation of Vir. is such that there exists a state that satisfies:

$$L_0|\epsilon> = \epsilon|\epsilon>; \qquad L_n|\epsilon> = 0; \qquad n > 0$$

ϵ is the highest weight. Gervais and Neveu have shown that (3.8) is the classical equivalent of the last equation, and that the solutions (3.6) does lead upon quantization to highest-weight states. Hence they are called highest-weight or ground-state solutions. The parameter λ determines the value of L_0. Classically it may take all values such that the value of L_0 is positive, that is $-1 < \lambda < 1$. In this interval the solution is regular everywhere when transformed by $z = \exp(\xi + i\eta)$ to the strip $0 < \eta < \pi$:

$$\begin{aligned} e^{-\phi/2} = \frac{1}{\sqrt{2}\lambda} \Bigg(& \operatorname{cotan}\left(\frac{\pi\lambda}{2}\right) \cosh(\lambda(\xi - \Delta)) + \\ & + \frac{\cos(\lambda(\eta - \pi/2))}{\sin(\pi\lambda/2)} \Bigg) \end{aligned} \tag{3.9}$$

It has become clear, by now, that the parameter Δ corresponds to the possibility of performing an arbitrary translation in the euclidean time ξ. Moreover, since the dependance in η is through trigonometric functions of $\lambda\eta$, the solution is multivalued when extented to the full cylinder $0 < \eta < 2\pi$. Thus the two points $z = 0$ and $z = \infty$ are

punctures on the resulting surface with constant curvature. This terminates the present discussion of the classical ground states.

As a general remark, let us say that, for $\lambda = \pm 1$, formula (3.9) reduces to

$$e^{-\phi/2} = \frac{1}{\sqrt{2}} \sin(\eta) \tag{3.10}$$

up to the jacobian of the mapping of the upper half plane, this exactly coincides with the metric of the Poincaré half plane. It is a limiting case of our class of metrics, however, since for $\lambda = \pm 1$, the residue of the double pole is equal to zero. In the language of conformal field theory, the corresponding highest weight state with weight zero is the SL (2c)-invariant vacuum.

Before turning to the case of several particles, we recall, following [3], that there is an infinite tower of harmonic excitations on top of the ground (highest-weight) states, that are different one-particle states. At the quantum level they correspond to the other states of the representations of Vir., besides the above highest-weight ones. The potentials are such that the Virasoro generators (3.8) have more than just a double pole. Consider, for instance, the case of two non-vanishing Fourier modes $(L_0, L_{\pm n})$:

$$U^{(n)}(z) = -\frac{1}{4}\frac{1-\lambda^2}{z^2} + \frac{\rho}{z^{2+n}} + \frac{\rho}{z^{2-n}} \tag{3.11}$$

One case higher poles than before. The associated solutions $\exp(-\phi/2)$ give another class of constant-curvature metrics on the sphere with two punctures that describe the propagation of the corresponding states.

4. P PUNCTURES – THE CLASSICAL SCATTERING THEORY OF P HIGHEST-WEIGHT STATES

In the general case, the Riemann surface has P branch points. Choosing the nature of the singualarities of U determines the quantum number of the particles which are scattered. This is seen by mapping a small neighborhood of each singular point on the cylinder that describes tha asymptotic free-particle states which we just discussed. In doing so, one must keep in mind that the potential U transforms in a non-trivial way under conformal transformation [3]:

$$U(z) \Rightarrow \left(\frac{\mathrm{d}w}{\mathrm{d}z}\right)^2 U(w(z)) - \frac{1}{2}\mathcal{D}_z(\omega); \quad \text{for} \quad z \Rightarrow w(z) \tag{4.1a}$$

where $\mathcal{D}$ is the Schwarzian derivatives:

$$\mathcal{D}_z(\omega) = \frac{\mathrm{d}^3\omega}{\mathrm{d}z^3}\left(\frac{\mathrm{d}\omega}{\mathrm{d}z}\right)^{-1} - \frac{3}{2}\left(\frac{\mathrm{d}^2\omega}{\mathrm{d}z^2}\right)^2\left(\frac{\mathrm{d}\omega}{\mathrm{d}z}\right)^{-2} \tag{4.1b}$$

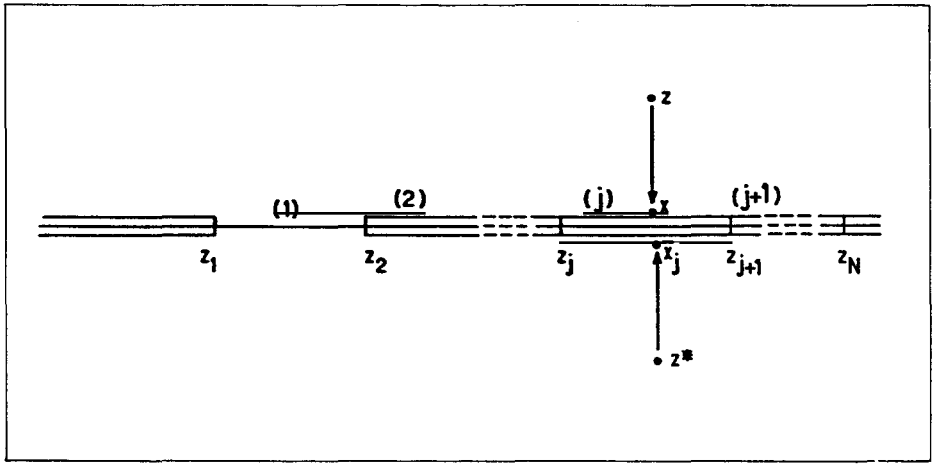

Fig. 1. The locations of the $P-1$ cuts and of the P branch points on the real axis.

By this, double poles of U are turned into zero Fourier-coefficients in the corresponding local tubes. Thus is we want to describe scattering of the highest-weight states discussed above, we should choose a potential U with double poles at any of the z_j, $1,\ldots,P$. In order to make contact with the solution of the previous section asymptotically, we specify the double pole so that, when z approaches any of the singular points z one has:

$$U(z) \sim -\frac{1}{4}\frac{1-\lambda_j^2}{(z-z_j)^2} \tag{4.2}$$

(If there are singularities at $z=\infty$, the asymptotic behavior should be such it takes the last form after one has mapped the infinity to a point at finite distance by, say, $z \to 1/z$, taking (4.1) into account). Equation (4.2) ensures that, when z approaches any of the singular points, the potential U takes the same form as for the one-highest-weight state.

The general features of the solutions $\psi_{1,2}$ follow from the standard monodromy theory of differential equations with meromorphic coefficients [1]. Around each singular point, the dominant term is the same as for the one-particle case. Hence, for each j there exists two solutions of the Schrödinger equation such that

$$\psi_{\frac{1}{2}}^{(j)}(z) \sim c_j \lambda_j^{-1/2} z^{\frac{1}{2}(1\pm\lambda_j)} \quad \text{as} \quad z \to z_j \tag{4.3}$$

where the c_j are appropriate constants. Clearly these solutions have no cut between z_j and z_{j+1}. They correspond to an analytic continuation to the sheet obtained by going through this interval. With this choice of basis in the space of solutions, the monodromy matrix is diagonal around the branch point z_j. This sheet is not the physical one, however, except for $j=1$, (see fig. 1) and one determines the monodromy matrices M of equation (2.14) by performing the analytic continuation that brings us back to it.

The classical scattering theory is straighforwardly developed, once the classical solution has been determined in such a way that it satisfies all boundary conditions. When z tends to z_j, we may let

$$z \sim z_j + e^{\xi_j + i\eta_j}; \quad \xi_j \to -\infty \tag{4.4}$$

The parameter ξ_j plays the role of the euclidean time, and this limit is the usual large-time limit of scattering theory. From (4.2) and (4.3), it next follows that the classical solution will tend to the one-ground-state classical solution (3.9), with $\lambda = \lambda_j$, and with a parameter Δ_j that is completely determined. This parameter is the so-called time delay of classical scattering theory. The novel point here, as compared with potential theory, is that we shall have P time delays, one for each singular point.

Let us remark that the physics is independent of the choice of what we call the physical sheet. Starting from a different sheet would just be equivalent to performing an overall Möbius transformation that is irrelevant. In particular, all the branch points are on the same footing and the result is invariant under circular permutations of them.

5. THE CLASSICAL SOLUTION FOR THE SCATTERING OF THREE HIGHEST-WEIGHT STATES

In the case of three puncture, we may use projective transformations to locate them at:

$$z_1 = 0; z_2 = 1; z_3 = \infty \tag{5.1}$$

The suitable potential is given by:

$$U(z) = -\frac{1-\lambda_1^2}{4z^2} - \frac{1-\lambda_2^2}{4(1-z)^2} - \frac{1-\lambda_1^2-\lambda_2^2+\lambda_3^2}{4z(1-z)} \tag{5.2}$$

This is the only possible choice that leads to double poles at 0, 1 and ∞, as one easily verifies making use of (4.1).

The solution of the scattering problem involving three asymptotic states is given by the solution of the Schrödinger equation (2.8) with the potential (5.2). It has three poles of second order, and the solutions of the Schrödinger equation with this potential are well known [12]. They are related to standard hypergeometric functions [12]:

PROPOSITION. *The following three pairs of functions are independent solutions of equation (2.8) that have diagonal monodromy matrices, when z circles around $z = 0$, $z = 1$,*

and $z = \infty$ respectively:

$$
\begin{aligned}
&\psi_1^{(1)}(z) = \lambda_1^{-1/2}\psi(\lambda_1,\lambda_2,\lambda_3;z);\\
&\psi_2^{(1)}(z) = \lambda_1^{-1/2}\psi(-\lambda_1,\lambda_2,\lambda_3;z)\\
&\psi_1^{(2)}(z) = i\lambda_2^{-1/2}\psi(\lambda_2,\lambda_1,\lambda_3;1-z);\\
&\psi_2^{(2)}(z) = i\lambda_2^{-1/2}\psi(-\lambda_2,\lambda_1,\lambda_3;1-z)\\
&\psi_1^{(3)}(z) = \frac{i}{z}\lambda_3^{-1/2}\psi(\lambda_3,\lambda_2,\lambda_1;\frac{1}{z});\\
&\psi_2^{(3)}(z) = \frac{i}{z}\lambda_3^{-1/2}\psi(-\lambda_3,\lambda_2,\lambda_1;\frac{1}{z})
\end{aligned}
\tag{5.3}
$$

where the ψ 's are related to the hypergeometric functions $F(a,b,c,z)$ by:

$$
\begin{aligned}
&\psi(\lambda,\mu,\rho;z) = z^{\frac{1}{2}(1+\lambda)}\,(1-z)^{\frac{1}{2}(1-\mu)}\\
&F(\frac{1}{2}(1+\lambda-\mu+\rho),\frac{1}{2}(1+\lambda-\mu-\rho),1+\lambda;z)
\end{aligned}
\tag{5.4}
$$

Proof. This a standard result discussed, for instance in ref. [12]. $\psi_{1,2}^{(1)},\psi_{1,2}^{(2)},\psi_{1,2}^{(3)}$, correspond to the three pairs $(u_5,u_1),(u_6,u_2),(u_3,u_4)$ of Kummer solutions. The monodromy property trivially follows from the analyticity of $F(a,b,c;z)$ at $z=0$. Q.E.D. ∎

Our conventions are such that the upper index i of $\psi_{1,2}^{(i)}$ characterizes the pair of independant solutions chosen, while the lower index m distinguishes the two solutions of a given pair. The prefactors $\lambda_i^{-1/2}$ are introduced in order that the Wronkian be equal to one.

PROPOSITION. *There exist unimodular 2×2 matrices $S_{mn}^{(ij}$ such that*

$$
\psi_m^{(i)}(z) = \sum_n S_{mn}^{ij}\psi_n^{(j)}(z) \tag{5.5}
$$

In this identity and in the following, no summation over repeated indices is to be carried out unless explicitely indicated.

Proof. The different $\psi^{(i)}$ are pairs of solutions of the same differential equation, and hence related by linear transformations. The matrix elements $S_{mn}^{(ij)}$ can be read off in

ref. [12] from the relations between Kummer solutions. They are expressed in terms of ratios of products of Γ-functions and phase factors. The explicit forms are given in ref. [1]. Q.E.D. ∎

From now on, the relevant quantities are given an upper index indicating with which pair of solution (5.3) they are defined.

PROPOSITION. *The solution (2.10) may be written in the three equivalent forms:*

$$e^{-\phi/2} = \frac{1}{2i\sqrt{2}} \left(\psi^{(j)}(z)\right)^T \zeta \chi^{(j)}(z^*) \tag{5.6}$$

where j *and* $1, 2$ *or* 3 *and*

$$\chi_m^{(i)}(z) = \sum_n S_{mn}^{(ij)} \chi_n^{(j)}(z) \tag{5.7}$$

Proof. This follows from explicit form of the ζ matrix and from the fact that the determinants of the matrices M are equal to one. Q.E.D. ∎

PROPOSITION. *Let* $M_j^{(i)}$ *be the monodromy matrix that describes the behaviour of the solutions* $\psi^{(i)}$ *when* z *goes once around the branch point* $z^{(j)}$ *counterclockwise*

$$\psi_m^{(i)}(z) \to \sum_n (M_j^{(i)})_{mn} \psi_n^{(i)}(z) \tag{5.8}$$

It is given by:

$$M_j^{(i)}(z) = S^{(ij)} \, M_j^{(j)} S^{(ji)} \tag{5.9}$$

where $M_i^{(i)}$ *is the diagonal matrix:*

$$M_i^{(i)} = -\begin{pmatrix} \exp(i\pi\lambda_i) & 0 \\ 0 & \exp(-i\pi\lambda_i) \end{pmatrix} \tag{5.10}$$

Proof. The result trivially follows from the above definition of the matrices $S_{mn}^{(ij)}$ (the explicit formulae can be found in reference [1]). ∎

PROPOSITION. *The matrices* $T_j^{(1)}$ *of equation (2.12) which relates* $\chi^{(1)}(\bar{x}^j)$ to $\psi^{(1)}(x)$ across the segment j of the real axis {segment 1: (0, 1), segment 2: $(1,\infty)$, segment 3: $(-\infty,0)$} are connected by

$$T_2^{(1)} = T_1^{(1)}\, M_2^{(1)}, \qquad T_3^{(1)} = T_1^{(1)}\,(M_1^{(1)})^{-1} \tag{5.11}$$

Proof. This an easy consequence of the above discussion. ∎

PROPOSITION. *Let us write:*

$$T_1^{(1)} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \tag{5.12}$$

Equations (5.10) and conditions (2.13), that is:

$$tr\, T_1^{(1)} = trT_2^{(1)} = trT_3^{(1)} = 2 \tag{5.13}$$

have the unique solution:

$$\alpha = -i\frac{e^{i\pi\lambda_1/2}}{\sin\frac{\pi\lambda_1}{2}}, \quad \delta = \alpha^*$$

$$\begin{aligned} \beta_\pm &= \frac{1}{(M_2^{(1)})_{21}}\frac{x_\pm}{2\left(\sin\frac{\pi\lambda_1}{2}\right)^2}; \\ \gamma_\pm &= \frac{1}{(M_2^{(1)})_{12}}\frac{x_\mp}{2\left(\sin\frac{\pi\lambda_1}{2}\right)^2} \end{aligned} \tag{5.14}$$

where we have introduced:

$$x_\pm = 2\left(\cos\frac{\pi\lambda_3}{2} \pm \sin\frac{\pi(\lambda_1+\lambda_2)}{2}\right)\left(\cos\frac{\pi\lambda_3}{2} \pm \sin\frac{\pi(\lambda_1-\lambda_2)}{2}\right) \tag{5.15}$$

Proof. The computation is straightforward. Details are given in [1]. ∎

We now come to our main formula for the three-highest-weight case:

THEOREM. *When z tends to z_j the three-highest-weight classical solution behaves, up to the Jacobian of the mapping to the local tube, according to:*

$$e^{-\phi/2} \sim \frac{1}{\sqrt{2}\lambda_i}\left(\operatorname{cotan}\frac{\pi\lambda_i}{2}\cosh(\lambda_i(\xi-\Delta_i)) + \frac{\cos(\lambda_i(\eta_i-\frac{\pi}{2}))}{\sin\frac{\pi\lambda_i}{2}}\right) \tag{5.16}$$

$$\text{for } z \to z_i,\ i = 1,2,3$$

where we have let:

$$z = e^{\xi_1+i\eta_1};\quad z = 1-e^{\xi_2-i\eta_2};\quad z = e^{-\xi_3+i\eta_3};\quad 0 \le \eta_1 \le \pi \tag{5.17}$$

when z tends to 0, and ∞ respectively. These formulae coincide with the one-highest-weight solution (3.9). The time delays Δ_i are given by:

$$\begin{aligned} F_i(\lambda_1,\lambda_2,\lambda_3) &\equiv e^{\lambda_i\Delta_i} = \\ &= \frac{\zeta(1-\lambda_i)\zeta(-\lambda_i)}{\zeta(1+\lambda_i)\zeta(\lambda_i)} \prod_{\epsilon_1,\epsilon_2,\epsilon_3=\pm} \zeta\left(\frac{1}{2}+\sum_{k=1}^{3}\frac{\epsilon_k\lambda_k}{2}\right)^{\epsilon_i} \end{aligned} \tag{5.18}$$

Proof. For $z \to z_i$, one uses (5.6) with $j = i$ since the functions $\psi_{(i)}$ and $\chi^{(i)}$ have a simple power law behaviour in the neighborhood of z_i. The computation of the time delays in simpler at $z = 0$, since the above formulae naturally involved $T_j^{(1)}$. One obtains:

$$e^{\lambda_1\Delta_1} = i\beta/\operatorname{cotan}\frac{\pi\lambda_1}{2} \tag{5.19}$$

Formulae (5.14, 15) give explicitely:

$$\begin{aligned} \lambda_1\Delta_1 = &-\ln\pi^2 - \ln\Gamma(-\lambda_1) + \ln\Gamma(\lambda_1) + \\ &+ \sum_{\epsilon,\epsilon'=\pm}\ln\Gamma(\frac{1}{2}(1-\lambda_1+\epsilon\lambda_2+\epsilon'\lambda_3)) + \\ &+ \sum_{\epsilon=\pm}\ln\left(\cos\frac{\pi\lambda_3}{2} \pm \sin\frac{\pi}{2}(\lambda_1+\epsilon\lambda_2)\right) \end{aligned} \tag{5.20}$$

This last formula is next shown to be equivalent to (5.17) for $i = 1$, by making use of the basic relation between ξ – and Γ – functions:

$$\cos\left(\frac{\pi y}{2}\right)\Gamma(y) = \frac{(2\pi)^y}{2}\,\frac{\zeta(1-y)}{\zeta(y)} \tag{5.21}$$

One expects that the other two points will lead to similar formulae since all three punctures are on the same footing. This is shown in [1]. One uses equation (5.6) with $i = 2$, and 3. The quantities introduced here with upper indices (1) have to be transformed according to (5.5, 7, 9). Q.E.D. ∎

6. THE CLASSICAL INTEGRABILITY CONDITION

Up to now, we have explicitly determined the Liouville field ϕ for the case of three punctures and, hence, the constant curvature metric $\exp(\phi)\hat{g}$ on the corresponding Riemann surface. $\exp(-\phi/2)$ is a complicated bilinear form of the functions $\psi(z)$ and $\chi(z^*)$ of the hypergeometric type, see eq. (2.10). However, in any of the three asymptotic regimes, $z \to z_j$, corresponding to the large local-time limit, «far from the interaction region», the solution $\exp(-\phi/2)$ tends towards the one-ground state solution (3.6) or (3.9) (Riemann surface with only two points) with a level determined value for the parameter Δ_i, given by equation (5.17). This agrees with the particle-scattering interpretation already put forward. As mentioned above, the Δ_i are to be regarded as delays of the Euclidian local-times ξ_i. In classical mechanics, there is a general relationship between time delays and the classical phase shifts [13]. If our interpretation is correct, a similar connection should be true here. In order to motivate the forthcoming discussion we consider, for a while, the example of a simple non-relativistic particle, following closely ref. [13].

PROPOSITION. *In the classical theory of a particle moving in one space dimension under the influence of a time-independant potential, the time of flight necessary to go from an initial point q_i to a final point q_f of a trajectory with energy E, is given by:*

$$t_f - t_i = \frac{\partial S(q_f, q_i; E)}{\partial E}; \quad S(q_f, q_i; E) = \int_{q_i}^{q_f} p(q, E)\,\mathrm{d}q \tag{6.1}$$

where the derivative is computed for fixed q_i and q_f.

Proof. This is an elementary exercise of classical mechanics. One computes:

$$\begin{aligned} \frac{\partial S(q_f, q_i; E)}{\partial E} &= \int_{q_i}^{q_f} \frac{\partial p(q; E)}{\partial E}\mathrm{d}q = \\ &= \int_{q_i}^{q_f} \left(\frac{\partial H(p, q)}{\partial p}\right)^{-1} \mathrm{d}q = \\ &= \int_{q_i}^{q_f} \frac{\mathrm{d}q}{\dot{q}} = \int_{t_i}^{t_f} \mathrm{d}t \qquad \text{Q.E.D.} \end{aligned} \tag{6.2}$$

■

Assume that the potential tends to zero at infinity. For q_f (resp. q_i) going to $+\infty$ (resp. $-\infty$) the particle is free, and above time of flight has the same behavior as in the free case. This means that it diverges as:

$$t_f - t_i \sim (q_f - q_i)\sqrt{\frac{m}{2E}}; \quad q_i \to -\infty, q_f \to +\infty \tag{6.3}$$

where m is the mass of the particle. One defines the time delays as the limit of the difference between (6.1) and (6.3). This gives a convergent answer if the potential vanishes fast enough:

(6.4)
$$\Delta(E) = \lim \frac{\partial}{\partial E}\left(S(q_f, q_i; E) - (q_f - q_i)\sqrt{2mE}\right); \quad q_i \to -\infty, q_f \to +\infty$$

PROPOSITION. *The time delay is twice the derivative of the classical phase shift with respect to the energy.*

Proof. This classical phase shift is defined from the leading WKB solution to the Schrödinger equation

$$H(-i\frac{\mathrm{d}}{\mathrm{d}q}, q)\psi(q; E) = E\psi(q; E) \tag{6.5}$$

which is given by

$$\psi(q, q_i; E) = \exp\left(i\int_{q_i}^{q} p(x, E)\,\mathrm{d}x\right) \tag{6.6}$$

where the wave function is normalized by the condition:

$$\psi(q_i, q_i; E) = 0$$

The phase shift is defined by letting q_i going to $-\infty$ and q going ot $+\infty$. For the same reason as for the time delay, one introduces the free-particle wave function that gives the dominant large q behaviour:

$$\psi_0(q, q_i; E) = \exp(i(q - q_i)\sqrt{2mE}) \tag{6.7}$$

and the phase shift is defined by the condition:

$$\psi(q_f, q_i; E) \sim \psi_0(q_f, q_i; E)\, e^{2i\delta(E)}; \quad q_i \to -\infty, q_f \to +\infty \tag{6.8}$$

obtaining:

$$2\,\delta(E) = \lim \int_{q_i}^{q_f}\left(p(x, E) - \sqrt{2mE}\right)\mathrm{d}x \tag{6.9}$$

Finally, comparing this last equation with (6.4) one sees that:

$$\Delta(E) = 2\frac{\mathrm{d}\,\delta(E)}{\mathrm{d}E}; \quad \text{Q.E.D.} \tag{6.10}$$

■

The discussion just carried our also shows the basic principle of scattering theory; one recovers the free propagation asymptotically. This principle has been much stressed already.

Going back to our problem, it is clear that the Δ_i we have introduced do play the role of time delays. On the other hand, for the two-puncture case, (analogous to the particle in free motion), L_0 is the Hamiltonian, so that the weight ϵ_i which is its value computed from the classical solution is to be identified with the energy in the asymptotic region. We are thus led to the:

PRINCIPLE. *Each time delay Δ_i is the derivative of the classical phase shift respect to the weight ϵ_i in the channel i :*

$$\Delta_i = \frac{\partial}{\partial \epsilon_i} \delta(\lambda_1, \lambda_2, \ldots, \lambda_3) \tag{6.11}$$

Note that we do not put a factor of 2 as in (6.10). The reason is that the definition (6.9) of potential theory is, form our viewpoint the sum of two time delays, one for propagating from $-\infty$ to the interaction region, and the other for propagating from this region to $+\infty$. These would be more clearly separated if we had taken a more general potential with different limits at plus and minus infinity. Our situation is even more general since we have one time delay for each puncture. Since should be derivative of the same phase shift, and since

$$\epsilon_i = \frac{1}{8\hbar}(1 - \lambda_i^2) \tag{6.12}$$

we arrive at:

PROPOSITION. *The time delays should satisfy the integrability conditions:*

$$\frac{\partial(\lambda_i \Delta_i)}{\partial \lambda_j} = \frac{\partial(\lambda_j \Delta_j)}{\partial \lambda_i} \tag{6.13}$$

■

We have that this condition is satisfied in the case of three punctures, as we shall show next, but have not, so far, derived equation (6.11) in general. Since it is field theory, the Liouville theory involves an infinite number of degree of freedom. For this situation, Gervais and Sakita [14] have given the general WKB wave functional associated with a given classical solution. It will probability play the same role as (6.6) in the derivation of (6.11). The problem is, to some extent, similar to soliton scattering in two dimensions [15].

PROPOSITION. *The time delays (5.18) satisfy the integrability condition (6.13).*

Proof. Equations (5.18) shows that the time delays may be written under the form:

$$\lambda_i \Delta_i = A(\lambda_i) + \sum_{\epsilon_1, \epsilon_2 \epsilon_3} \epsilon_i B(u); \qquad u = \sum_{k=1}^{3} \epsilon_k \lambda_k \tag{6.14}$$

where A and B are functions of a single variable. Therefore:

$$\frac{\partial(\lambda_i \Delta_i)}{\partial \lambda_j} = \sum_{\epsilon_1, \epsilon_2 \epsilon_3} \epsilon_i \epsilon_j \frac{\mathrm{d}G}{\mathrm{d}u} \tag{6.15}$$

is symmetric in i and j. Q.E.D. ∎

Coming back to the physical picture, we finally note that the classical phase shift is given by the following three-dimensional line integral in $\vec{\lambda}$-space

$$\delta(\vec{\lambda}) = -\frac{1}{4\hbar} \int_{\vec{\lambda}_0}^{\vec{\lambda}} \sum_{i=1}^{3} \mu_i \Delta_i(\vec{\mu}) \mathrm{d}\mu_i, \qquad \vec{\lambda} = (\lambda_1, \lambda_2, \lambda_3) \tag{6.16}$$

Due to the integrability conditions we just verified, this line integral does not depend on the path chosen to go grom $\vec{\lambda}_0$ to $\vec{\lambda}$, but only on the starting and end points. The starting point $\vec{\lambda}_0$ may be arbitrarily chosen. Different choices give the same shift up to an irrelevant $\vec{\lambda}$-independent constant.

In scattering theory, the classical analogue of the S-matrix is just the exponential of the classical phase shift (see e.g. (6.8)). It is thus given by the exponential of an integral of logarithms. In the quantum case, the integral will be replaced by a discrete sum and the S-matrix will turn out to be given by a rather simple product. This is the issue to which we now turn.

7. THE EXACT QUANTUM S-MATRIX

It is a rather genearl feature of completely integrable field theories in $1 + 1$ dimensions that the exact quantum S-matrix is obtained from its quasi-classical approximation by relatively minor modifications. One of the first cases where this has been observed was the scattering of two sine-Gordon solitons. The many similarities between the sine-Gordon and the Liouville theories [3, 6] have been reviewed in ref. [1]. The genearl recipe for obtaining the quantum S-matrix from the classical one, which has proven to provide the correct answer for the sine-Gordon theory [16] and other integrable models [17] is the following:

1) Replace the coupling constant by a renormalized one.

2) Discretize the integral appearing in the classical S-matrix so as to allow only those values of the integration variables which correspond to the asymptotic quantum spectrum.

For the Liouville theory, the renormalized Liouville coupling constant is

$$\hbar_r = \hbar\eta^2 ; \eta = \frac{1}{4\hbar}(1 - \sqrt{1 - 8\hbar}) \tag{7.1}$$

There are two regimes. The so-called weak coupling regime is for $\hbar < 1/8$ where the renomarlized couping constant (7.1) is real. There, Gervais and Neveu [6] found a discrete set of integrable cases which are such that

$$\frac{1}{2\hbar_r} = N ; N \text{integer} > 0 . \tag{7.2}$$

or

$$\hbar = \frac{N}{2(N+1)^2} ; \ \eta = \frac{N+1}{N} \tag{7.3}$$

The present discussion, which starts from the classical Liouville dynamics is directly relevant to this weak coupling regime only.

Next, the classical S-matrix we determined in the last section (see eq. (6.9)) is given by the exponential of the classical phase shift

$$\exp\left(-\delta(\vec{\lambda})\right) \equiv \exp\left(\frac{1}{4\hbar}\int \sum_{i=1}^{3} \ln(F_i)\, d\mu_i\right) ; \ F_i = e^{\mu_i \Delta_i} \tag{7.4}$$

In order to go to the quantum case, we discretize the integral following the general philosophy outlined above. The continuous parameters $\vec{\lambda}$ are replaced by discrete ones. We shall only consider the integral cases where (7.3) holds. Then, we know from the exact quantum treatment of the free theory [6], that if we write the quantum version of eq. (4.6) as:

$$\epsilon_i = \frac{1}{8\hbar}\left(1 - \frac{\lambda_i^2}{\eta^2}\right) \tag{7.5}$$

the allowed values of λ_i are

$$\lambda_i = \frac{2m_i}{N} ; \ |m_i| \leq \frac{N-1}{2} \tag{7.6}$$

where m_i take integer (half integer) values if N is odd (even). The integer N is of course related to the Liouville coupling constant $\hbar$ through eq. (7.3).

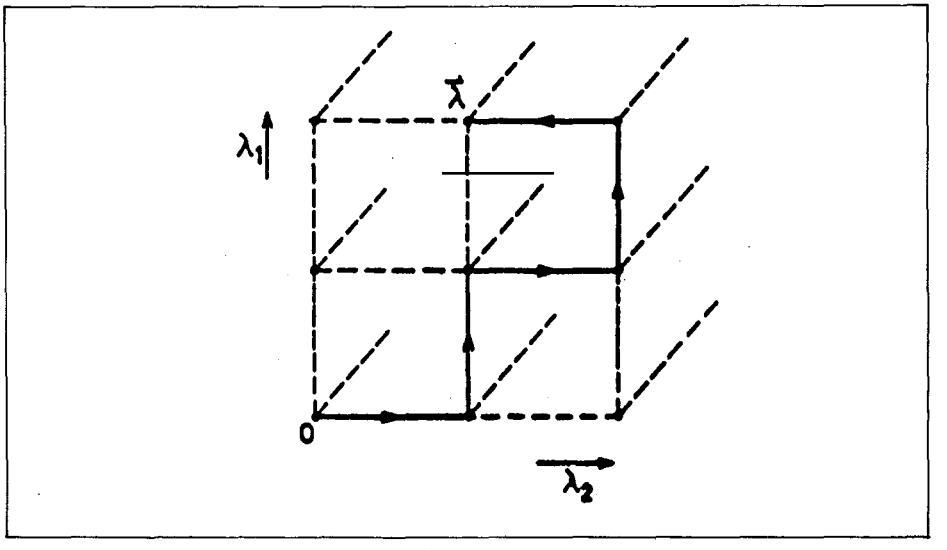

Fig. 2. The path on three-dimensional $\vec{\lambda}$-lattice corresponding to the expression given in eq. (5.19).

First, the coupling constant in front of the action (7.4) is to be replaced by the renormalized one which is given by (7.1, 2). This is quite natural, since at the classical level, the factor $1/4\hbar$ came from formula (6.12) which is now modified according to (7.5) where an additional η^2 factors appears. We now discretize eq. (7.4) by restricting the $\vec{\lambda}$ to lie on the cubic lattice defined by eq. (7.6). Remarkably, the product of the factor $1/4\hbar_r$ with the spacing $2/N$ of the $\vec{\lambda}$-lattice is equal to one, so that one ends up with producst of terms raised to the power plus or minus one. (A similar phenomenon already occured in the sine-Gordon theory.) The ansatz for the exact quantum S-matrix is

$$S(\vec{m}) = \prod_{\vec{n}=\vec{m}_0}^{\vec{m}} \left(\prod_{i=1}^{3} (F_i^{qu}(\vec{n})) \right)^{\epsilon_i}, \quad \epsilon_i = -1, 0, 1 \tag{7.7}$$

The set of allowed values (7.6) of $\vec{m}$ form a three-dimensional cubic lattice. $\vec{n}$ denotes a general point and $F_i^{qu}(\vec{n})$ is meant to be the quantum version of the corresponding classical quantity $F_i(\vec{\lambda})$ given by eq. (5.18). The product in eq. (7.7) needs some explanation: In the classical expression, the integration path started at $\vec{\lambda}_0$ and ended at the point in $\vec{\lambda}$-space that described the three asymptotic one-ground-state solutions. It is now replaced by a path on the $\vec{\lambda}$-lattice defined by eq. (7.6) (see fig. 2).

For each link of the path going in the direction i the product defining the S-matrix contains a factor F_i defined on the corresponding link. It is taken to the power $+1$ or -1 according to wether m_i increases or decreases along this link. This already suggest that the F_i should better be defined on the links rather than on the sites of the lattice.

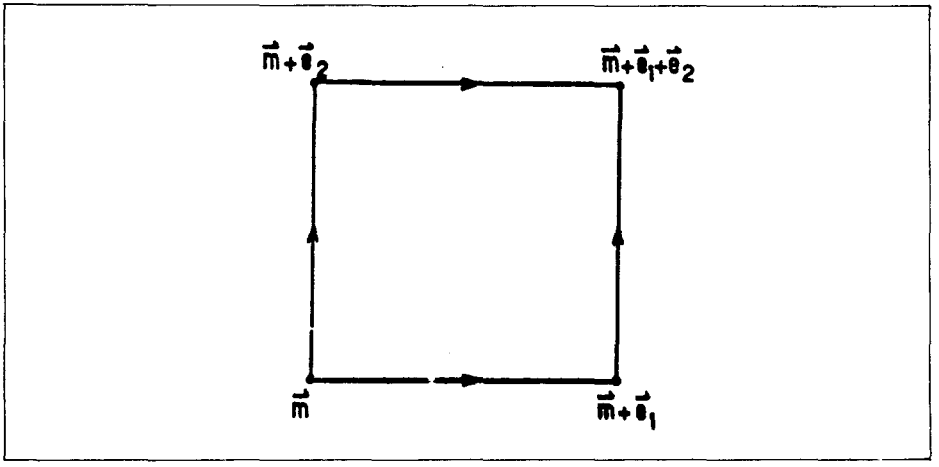

Fig. 3. The two paths around the elementary plaquette leading to the same result.

We conjecture that for a link going from $\vec{m}$ to $\vec{m}+\vec{e}_i$ ($\vec{e}_i$ being the unit vector in the direction i) the quantum version F_i^{qu} is but the classical expression of F_i (see equation (5.18)) taken at the value of $\vec{\lambda}$ corresponding to the midpoint of the link:

CONJECTURE. *The quantum S-matrix is given by the equation (7.7) with*

$$F_i^{qu}(\vec{m}) = F_i\left(\vec{\lambda} = 2\,\frac{\vec{m}+\frac{1}{2}\vec{e}_i}{N}\right) \tag{7.8}$$

where $F_i(\vec{\lambda})$ is given by equation (6.5).

If our prescription for the computation of the quantum S-matrix is to make sense, the result again must be independent of the path chosen to go from $\vec{m}_0$ to $\vec{m}$. Indeed, we have the following theorem:

THEOREM. *The quantum S-matrix (7.7) only depends on the endpoints $\vec{m}_0$ and $\vec{m}$, not on the path on the lattice chosen to join them.*

Proof. It is equivalent to demonstrate that the two paths shown in fig. 3 around the elementary plaquette on the lattice yield the same contribution, i.e. we have to show that

$$\begin{aligned} &F_i\left(2\,\frac{\vec{m}+\frac{1}{2}\vec{e}_i}{N}\right) F_j\left(2\,\frac{\vec{m}+\vec{e}_i+\frac{1}{2}\vec{e}_j}{N}\right) = \\ &= F_j\left(2\,\frac{\vec{m}+\frac{1}{2}\vec{e}_j}{N}\right) F_i\left(2\,\frac{\vec{m}+\vec{e}_j+\frac{1}{2}\vec{e}_i}{N}\right) \end{aligned} \tag{7.9}$$

For this, it is crucial that we took the argument of the F_i on the midpoints of the links. Any other choice would not lead to the necessary independence from the path and would be inconsistent. Using equation (6.14) it is trivial to check that equation (7.9) holds. ∎

In the classical limit, $N \to \infty$, the quantum S-matrix (7.7, 8) gives back the classical expression by construction.

Given the general succes of the discretization procedure on the one hand, and the fact the consistency condition (7.9) is satisfied on the other hand, we are very confident that the above conjecture is correct: eqs. (7.7, 8) give the exact quantum S-matrix for the interaction of three Liouville-heighest-weight states.

What is the geometrical meaning of this quantum S-matrix? In the quantum geometry on the Riemann surface, we are dealing with at present, the S-matrix gives the transition probababilty between the three asymptotic quantum geometries, labelled by $\lambda_1, \lambda_2, \lambda_3$, in the infinitesimal neighbourhoods of the branch points z_1, z_2, z_3.

8. CONCLUSIONS AND OUTLOOK

We have given the classical solution of the Liouville equation with the relevant boundary conditions for a Riemann surface with three branch points (two cuts) generalizing the earlier result of Gervais and Neveu for two branch points (one cut). While the latter corresponds to the propagation of one-highest-weight-solutions, the former describes the interactions (scattering) of three of these ground-states. In both cases this leads to one – and three – parameter families of constant curvature metrics on the Riemann surfaces.

The classical scattering theory (for three asymptotic states: three branch points) was discussed and the time delays were derived. These were shown to satisfy an intriguing zero-curl condition, which physically was to be expected, but whose mathematical significance remains to be elucidated. Finally we discussed the quantum scattering and the quantum S-matrix was conjectured. These results are relevant to the quantum geometry of string theory [1].

We have not discussed the generalization to more than there branch points. This would be based on the monodromy properties of the solutions of second order differential equations with more than three singular points, which is much more difficult problem mathematically.

Finally let us mention that the Liouville theory is a member of the family of so-called Toda theories. These are n-component generalizations of the Liouville theory, associated with a rank n simple Lie algebra. As is the case for the Liouville theory, the Toda theories are completely integrable and conformally invariant. They furthermore exhibit extended (higher-spin) Virasoro symmetries. We have recently investigated them in details [18] and shown that they yield a systematic approach to higher-spin conformal theories. In principle, the discussion of the present article may be repeated for the scattering

of three Toda ground-states. However, for the moment it would be more important to obtain a geometrical interpretation of the Toda fields, similar to the constant-curvature metric condition which has proven to be very fruitful in the present Liouville case.

REFERENCES

[1] A. BILAL, J. L. GERVAIS: *Exact Quantum Three-point Function of Liouville Highest-Weight States*, Nucl. Phys. B305 [FS23] (1988) 33.

[2] B. DURHUUS, P. OLESEN, J. L. PETERSEN: *Polyakov's Quantized String with Boundary Terms*, Nucl. Phys. B198 (1982) 157.

[3] J. L. GERVAIS, A. NEVEU: *Dual String Spectrum in Polyakov's Quantization* (I), Nucl. Phys. B199 (1982) 59; (II), Nucl. Phys. B209 (1982) 125.

[4] A. BILAL: *PhD thesis*, Ecole Normale Supérieure, Paris, 1988.

[5] A. M. POLYAKOV: *Quantum Geometry of Bosonic String*, Phys. Lett. 103B (1981) 207.

[6] J. L. GERVAIS, A. NEVEU: *New Quantum Treatment of Liouville Field Theory*, Nucl. Phys. B224 (1983) 329; *Novel Triangle Relations and Absence of Tachyon in Lioville String Theory*, Nucl. Phys. B238 (1984) 1983.

[7] J. L. GERVAIS, A. NEVEU: *New Critical Dimensions: for String Theories*, Nucl. Phys. B284 (1987) 397; *Modular Invariance for Closed Strings at the New Critical Dimensions*: Phys. Lett. B187 (1987) 1; *The Five Dimensional Open Liouville Superstring*: Nucl. Phys. B293 (1987) 1; *Liouville Superstring and Ising Model in Three Dimensions*: Nucl. Phys. B295 (1988) 277; for a rievew see J. L. Gervais *Liouville superstring* Proceeding of the International Conference on Conformal Field Theory and Related Topics in *Conformal Invariance and String Theory* Academic Press 1988.

[8] B. M. WILSON: *Proceeding of the London Mathematical Society* (1923) 25.

[9] S. RAMANUJAN: *Messenger of Mathematics*, 45 (1915) 81.

[10] A. ERDELY, W. MAGNUS, F. OBERHETTINGER, F. G. TRICOMI: *Higher Transcendental Functions*, Bateman proj., 1955, vol. 3.

[11] see, e.g., E. POOLE: *Introduction to the Theory of Linear Differential Equations*, Oxford University press 1936.

[12] A. ERDELYI, W. MAGNUS, F. OBERHETTINGER, F. G. TRICOMI: *Higher Transcendental Functions*, Bateman proj., 1955, vol. 1.

[13] R. JACKIW, G. WOO: *Semiclassical Scattering of Quantized Nonlinear Waves*, Phys. Rev. D12 (1975).

[14] J. L. GERVAIS, B. SAKITA: *WKB Wave Function for Systems with Many Degrees of Freedom : A unifed View of Solitons and Pseudoparticles*, Phys. Rev. D16 (1977) 3507; H. J. DE VEGA, J. L. GERVAIS, B. SAKITA: *Real-Time Approach to Instanton Phenomena*, (I) Nucl. Phys. B139 (1978) 20; (II) Nucl. Phys. B142 (1978) 125.

[15] J. L. GERVAIS, A. JEVICKI: *Quantum Scattering of Solitons*, Nucl. Phys. B110 (1976).

[16] L. D. FADDEV, P. P. KULISH, V. E. KOREPIN: JETP Pisma 21 (1975) 302; L. D. FADDEV, I. A. S. preprint (1975).

[17] A. B. ZAMOLODCHIKOV: JETP Pisma 25 (1977) 499; M. KAROWSKI, H. J. THUN: *Complete S-matrix of the Massive Thirring Model*, Nucl. Phys. B130 (1977) 295.

[18] A. BILAL, J. L. GERVAIS: *Systematic Approach to Conformal System with Extended Virasoro Symmetries*, Phys. Lett. 206B (1988) 412; *Extended $C = \infty$ Conformal Systems from Classical Toda Field Theories*, Nucl. Phys. B314 (1989) 646, *Systematic Construction of Conformal Theories with Higher-Spin Virasoro Symmetries*, Nucl. Phys. B318 (1989) 579.

Manuscript received: November 15, 1988